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## The Intermediate Coupling Regime in the AdS/CFT Correspondence

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### Abstract

The correspondence between the 't Hooft limit of  $N = 4$  super Yang-Mills theory and tree-level IIB superstring theory on  $\text{AdS}_5 \times S^5$  in a Ramond-Ramond background at values of  $\lambda = g^2 N$  ranging from infinity to zero is examined in the context of unitarity. A squaring relation for the imaginary part of the holographic scattering of identical string fields in the two-particle channels is found, and a mismatch between weak and strong 't Hooft coupling is pointed out within the correspondence. Several interpretations and implications are proposed.

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# 1 Introduction

Recently there has been much interest in understanding the 't Hooft limit of  $N = 4$  super Yang-Mills theory at strong coupling via the conjectured correspondence with IIB superstring theory compactified on  $AdS_5 \times S^5$  in a non-vanishing Ramond-Ramond background [1, 2, 3, 4]. Despite the developments in technology useful in computing correlation functions at large  $\lambda$  through the classical holographic supergravity description, there has been little work on interpolating correlation functions beyond three-point between the two regimes. As the four-point functions are known not to correspond to the free-field limit at large coupling, this would involve resumming an infinite set of planar Feynman diagrams and comparing with the scattering element derived from the infinite number of string exchanges in tree-level string theory.

Both these calculations appear hard in that summing an infinite number of Feynman diagrams seems intractable, while the techniques of perturbative string theory on  $AdS_5 \times S^5$  in a Ramond-Ramond background have not been fully developed; progress along the direction in [5, 6] would be needed. The general four-point function at string tree level in the anti-de Sitter (RR) background is a function  $F(\lambda, k_i)$  of the 't Hooft coupling  $\lambda$ . At large  $N$  and finite  $\lambda = g^2 N$ , scattering elements obey a factorization condition inherited from the classical field equations [7]; for those between identical fields the factorization condition may be used to obtain a non-negativity condition on the imaginary part of four-point functions in the two-particle channels. On the other hand, the result at  $\lambda = 0$  in  $N = 4$  super Yang-Mills theory for these dual correlators is easily computable at four-point (in  $x$ -space as well as in  $k$ -space). We shall combine these two calculations in the different limits to probe the conjectured correspondence at finite 't Hooft coupling and at large  $N$ .

Extending the weakly coupled string theory calculation to the regime of strong coupling may invalidate the classical approximation (the anti-de Sitter space is preserved under quantum corrections to leading order corrections in the IIB effective action given in [8]). However, it is of interest to find potential properties of IIB holographic string theory on AdS that may lift to the dual boundary field theory in the intermediate coupling regime (i.e.  $\lambda \sim 1$ ). There are several forms of the conjectured correspondence between IIB string theory on  $AdS_5 \times S^5$  in a background RR field and  $N = 4$  super Yang-Mills theory; we note beforehand that further non-perturbative processes are in accord with summing over different asymptotically AdS spacetimes. Our intention here is to examine the holographic scattering in the spirit of the Gross-Mende effect [9] (limiting to a tensionless  $R^2/\alpha' \rightarrow 0$  string).

Rather than computing the scattering element on  $AdS_5 \times S^5$  in a constant Ramond-Ramond background, we shall use a unitarity argument to find a positivity condition on the imaginary part in the two-particle channel for particular correlator examples where all of the external fields are of the same type. We assume

that unitarity is sensible in the bulk theory, and we use the  $i\varepsilon$  prescription provided in [7], following from the naive Wick rotation of the Euclidean geometry. (See [10] and [11] for a detailed account of the  $i\varepsilon$  in critical and compactified four-point string amplitudes.) Unitarity has been examined in detail in [12, 7, 13]. In the following, we define  $s = s_{12}$ ,  $t = s_{23}$  and  $u = s_{13}$ ; then  $s+t+u = \sum k_j^2$ . Holding  $k_j^2 \leq 0$  together with  $t, u < 0$  and  $s > 0$  we extract the imaginary parts in the s-channel by complex conjugation [7]. In conventional perturbative string theory on a curved space at any finite value of the  $s$  value only a finite number of Kaluza-Klein and string states may contribute to the spectral density, and hence to the imaginary parts at finite  $s$ . In the holographic formulation an infinite number of string modes contribute to the imaginary parts. The analysis performed here relies only on the existence of field equations for the modes of the string on the compactified space, thus avoiding the technical complications associated with a path integral quantization. We further assume that a consistent truncation exists for the field equations coming from the compactified string theory; this has been shown for the KK compactification of eleven dimensional supergravity in an  $AdS_4 \times S^7$  background [14] and recently in the  $AdS_7 \times S^4$  context [15]; although this has not been shown technically in the case at hand, we shall assume that it exists.

The cut in the  $s$ -channel, if we choose the external lines to be of the same particle type, is of the form

$$\text{Im}_s A_{ffff}(k_j) = \sum_i M_i(k_1, k_2; k_1 + k_2) M_i(k_3, k_4; k_3 + k_4) , \quad (1.1)$$

after integrating over the bulk (fifth) holographic coordinate perpendicular to the boundary. The sum over  $i$  extends over all contributing intermediate states in the particular holographic four-point function. We now further examine the scalar contributions. The factorization condition follows from decomposing the bulk-bulk propagator through the relation

$$G(x, y) = - \sum_n \frac{\phi_n^*(x) \phi_n(y)}{\lambda_n^2 - i\varepsilon} , \quad (1.2)$$

where  $\phi_n(x)$  span a complete set of eigenfunction solutions of the kinetic operator (with appropriate boundary conditions) for the particular field we are considering. For example, the propagator for massive scalar fields has the form after a Wick rotation (in Minkowski anti-de Sitter Poincare coordinates  $ds^2 = 1/x_0^2(dx_0^2 + d\vec{x}^2)$  where we have the boundary four-dimensional metric given by  $u \cdot v = u_0 v_0 - \vec{u} \cdot \vec{v}$ ),

$$G(x, y) = -i \int_0^\infty d\lambda \lambda \int \frac{d^d k}{(2\pi)^d} \frac{\varphi_\lambda^*(x) \varphi_\lambda(y)}{(\lambda^2 - \vec{k}^2 - i\varepsilon)} , \quad (1.3)$$

and obeys

$$\hat{K} G(x, y) = -i x_0^{d+1} \delta^{d+1}(x - y) = -i \frac{\delta^{d+1}(x - y)}{\sqrt{g}} , \quad (1.4)$$

where

$$\hat{K} = - \left( \frac{1}{\sqrt{g}} (\partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu) - m^2 \right). \quad (1.5)$$

The eigenfunctions  $\varphi_\lambda(y)$  obeying the correct Dirichlet boundary conditions at  $x_0 = 0$  are

$$\begin{aligned} \varphi_\lambda(x) &= x_0^{d/2} e^{i\vec{k}\cdot\vec{x}} J_\nu(\lambda x_0), \quad \vec{k}\cdot\vec{x} \equiv \sum_{i=1}^d k_i x^i \\ \hat{K}\varphi_\lambda(x) &= -(\lambda^2 - \vec{k}^2) x_0^2 \varphi_\lambda(x), \end{aligned} \quad (1.6)$$

where  $\nu = \sqrt{m^2 + d^2/4} > 0$ . In Poincare coordinates  $\varphi_\lambda(x)$  are labelled by the four-vector  $\vec{k}$ , the conserved momentum along the boundary, and a continuous eigenvalue  $\lambda$ . The  $i\varepsilon$  prescription is chosen to agree with that on the boundary  $N = 4$  super Yang-Mills theory via a Wick rotation. The result for the correlation function will be proportional to a factor of  $i$ , both in the string theory and in the boundary theory; as we are exploring the analytic properties of the correlation function we shall drop the factor of  $i$  in (1.3) as well as in the four-point function.

Further modifications of the Greens functions in Minkowski space via adding normalizable zero modes to them [12, 16] will not alter the factorization of the general four-point function, but rather change the functional form of  $M_i$  in (1.1) by the addition of terms to the propagators. Although we have reproduced the explicit form of the massive propagator in (1.3), the factorization condition found from the complete set of states in (1.2) is expected to hold for general fields with spin as does in perturbative string theory.

Holding  $k^2 < 0$  in (1.3) extracts the imaginary part within the  $k$ -integration (via  $\text{Im}(\lambda^2 - \vec{k}^2 - i\varepsilon)^{-1} = -\pi\delta^{(d)}(\lambda^2 - \vec{k}^2)$ ) for the propagating modes in the interior of the anti-de Sitter spacetime. This imaginary part reproduces the factorization formula in (1.1).

Note that, for an arbitrarily massive mode, the imaginary part picks up a contribution from a small negative value of  $k^2$ , as is clear from (1.3) and (1.6). Each term in the sum in (1.1) gives rise to a power series expansion in  $\lambda$ , from expanding the mass dependence in the integrals over the bulk-boundary-boundary three-point functions (the argument of the Bessel function depends on the mass). The fact that an infinite number of states contribute to the unitarity cut at finite  $s$  is a holographic feature differing from usual perturbative string theory; the latter gives contributions in accord with particle thresholds; i.e.  $1/k^2 - m^2 + i\varepsilon$  in propagating states. However, at finite  $\lambda$ , where one is effectively extending the analysis on the field theory side to an infinite number of Feynman diagrams, this is in accord with extracting unitarity cuts in two-particle channels of multi-loop Feynman diagrams of arbitrary loop order. In each order within a loop expansion in perturbation theory of the correlation function, one expects a contribution of multi-particle cuts in the two-particle channel (for example, the double-box Feynman diagram contributes both

two- and three-particle cuts). This holographic unitarity feature appears to accord with expectations from field theory although the positivity condition is not clear in field theory.

This factorization of the holographic  $S$ -matrix elements at tree-level was noted in [7], and follows from the fact that there is no multi-body phase space integration to be carried out at string tree-level on the anti-de Sitter space: it is a property of the classical string description of the large  $N$  limit of the gauge theory. Unitarity restricts the functions  $M_i$  to be real. Holding  $k_1 = k_4$  and  $k_2 = k_3$  (and other quantum numbers associated to these lines the same) we sum over all terms contributing on the right-hand side of the sum; each term contributes a square of a real function and thus must be positive (strictly, non-negative). However, we must be concerned with the presence of an infinite number of terms. For any finite number of terms the sum must be positive for any finite (positive) value of  $s$ . If the sum becomes negative after taking into account the infinite number of terms, then a pole or cut is evidenced, reflecting a finite radius of convergence.

The non-negativity condition does not depend on the detailed form of string perturbation theory in a Ramond-Ramond background, but rather on the use of field equations in a Ramond-Ramond background. We should note that bulk four-point vertices contribute to the imaginary parts at exceptional values of momenta for the massless fields, as at  $s = 0$ , but this limit does not enter into this analysis.

We now compare with the field theory predictions arising at the free limit, i.e.  $\lambda = 0$ . We normalize the protected gauge invariant operators,  $\text{Tr} \phi^{(i_1} \dots \phi^{i_k)}(z)$ , where the  $\text{SO}(6)$  vectors  $\phi^i$  are the  $N = 4$  scalars in the adjoint representation of  $\text{SU}(N)$ , in the form,

$$\mathcal{O}(z) = \frac{N}{(g^2 N)^{k/2}} \text{Tr} \phi^{(i_1} \dots \phi^{i_k)}(z) , \quad (1.7)$$

so that the free-field limit is independent of  $\lambda$  and proportional to  $N^2$ ; the Lagrangian has the microscopic fields suitably scaled to agree with the factor  $N^2$  of correlators of the gauge invariant composite operators in the free-field approximation. We are only considering the symmetrized traceless tensor product of scalar fields  $\phi^i(z)$  for simplicity.

In this normalization, the free-field result for the four-point function of symmetric bi-linear operators  $\mathcal{O}(z) = \frac{1}{g^2} \text{Tr} \phi^{(i} \phi^{j)}(z)$  is explicitly in  $x$ -space the box diagram and is proportional to  $N^2$ . In four dimensions, we have for the (unrenormalized) result, where the propagator is  $\Delta(z) = \frac{1}{4\pi^2} \frac{1}{z^2 - i\varepsilon}$ ,

$$\langle \prod_j \mathcal{O}^{m_j n_j}(z_j) \rangle = \frac{N^2}{(4\pi^2)^4} T^{\{mn\}} \left[ \frac{1}{(z_{12}^2 - i\varepsilon)(z_{23}^2 - i\varepsilon)(z_{34}^2 - i\varepsilon)(z_{41}^2 - i\varepsilon)} \right] \quad (1.8)$$

$$+ \frac{1}{(z_{12}^2 - i\varepsilon)(z_{13}^2 - i\varepsilon)(z_{34}^2 - i\varepsilon)(z_{42}^2 - i\varepsilon)} \quad (1.9)$$

$$+ \frac{1}{(z_{13}^2 - i\varepsilon)(z_{23}^2 - i\varepsilon)(z_{24}^2 - i\varepsilon)(z_{41}^2 - i\varepsilon)} \Big] , \quad (1.10)$$

where  $T^{\{mn\}}$  is the group theory factor associated with the single trace-operators and  $z_{ij}^2 = (z_i - z_j)^2$ . Similar results may be obtained regarding other free-field correlators in  $x$ -space.

We choose to obtain the result in  $k$ -space, where we may compare with previous results. After taking the Fourier transform, and for external momenta satisfying  $k_j^2 = 0$ , we obtain the following expression for the box diagram [17], with ordering of momenta at the vertices  $k_1 \dots k_4$ ,

$$I(s, t) = N^2 \frac{c(\epsilon)}{st} \left[ \frac{1}{\epsilon^2} ((-s)^{-\epsilon} + (-t)^{-\epsilon}) - \frac{1}{2} \ln^2(-s/-t) \right] . \quad (1.11)$$

Here,

$$I(s, t) = -i \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2(l - k_1)^2(l - k_1 - k_2)^2(l + k_4)^2} , \quad (1.12)$$

where, in including the appropriate  $i\varepsilon$ , we have to change  $s_{ij} \rightarrow s_{ij} + i\varepsilon$ . The normalization in (1.12) explicitly has a factor of  $-i$  to account for the removal of  $i$  in the dual string theory calculation (1.6). The dimension dependent constant  $c(\epsilon)$  is

$$c(\epsilon) = \frac{1}{2(4\pi)^2} \frac{\Gamma(1 + \epsilon)\Gamma^2(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} . \quad (1.13)$$

Note that by analytic continuation from higher than four dimensions the box diagram evaluated at zero momentum is formally zero, i.e.  $I^\epsilon(k_i = 0) = 0$ . We have also regulated the above integral using dimensional reduction, where  $d = 4 - 2\epsilon$ ; holding  $\epsilon < 0$  regulates the infra-red divergences appearing within the above integration. The integral is ultra-violet finite but keeping  $\epsilon > 0$  would have regulated this occurrence in any case. The complete expression for the four-point correlator in the free-field limit of bi-linear operators is the permuted sum of the integrals, i.e.  $I(s, t) + I(s, u) + I(t, u)$ .

The expansion in  $\epsilon$  of the particular (infra-red divergent) integral in (1.11) is explicitly

$$I(s, t) = N^2 \frac{c(\epsilon)}{st} \left[ \frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln(st) - \frac{1}{2} \ln(-s) \ln(-t) \right] . \quad (1.14)$$

The imaginary parts in the two-particle channels may be extracted from the expanded form in (1.14) via complex conjugation.

To end our discussion of the box integrations we give the most general case, which would hold for non-exceptional values of external momenta. The result for general momenta  $k_j^2 \leq 0$  is more complicated, but may be written down in terms of known functions as [17]

$$I^{4m} = \frac{1}{r(st - k_1^2 k_2^2)} \left[ \text{Li}_2\left[\frac{1}{2}(1 + r)\right] - \text{Li}_2\left[\frac{1}{2}(1 - r)\right] \right] \quad (1.15)$$

$$+\text{Li}_2[-\frac{1}{2\lambda}(1-2\lambda-r)] - \text{Li}_2[-\frac{1}{2\lambda}(1-2\lambda+r)] - \frac{1}{2}\ln(\lambda)\ln(\frac{1+r}{1-r}) , \quad (1.16)$$

where

$$r = [1-4\lambda]^{1/2}, \quad \frac{d}{dx}\text{Li}_2(1-x) = \frac{\ln x}{1-x} , \quad (1.17)$$

and,

$$\lambda = -\frac{k_1^2 k_3^2}{st} . \quad (1.18)$$

This box integral becomes infra-red divergent when we take any  $k_j^2 \rightarrow 0$ : soft for individual null momenta, and collinear for any two momenta flowing into adjacent vertices becoming null. If we maintain the dimensionally regularized form, keeping the full  $\epsilon$  dependence of the above (not shown), then the preceding box integral follows.

We now find the imaginary parts in the correlation functions in the free-field limit arising from the three contributing permuted box diagrams. Holding  $s > 0$  and  $t, u < 0$ , together with  $k_j^2 \leq 0$ , we obtain for the imaginary part,

$$\text{Im}_s I(s, t) = -N^2 \pi \frac{c(\epsilon)}{st} \left[ \frac{1}{\epsilon} + \frac{1}{2} \ln |t| \right] , \quad (1.19)$$

together with the  $t \leftrightarrow u$  contribution,

$$\text{Im}_s I(s, u) = -N^2 \pi \frac{c(\epsilon)}{su} \left[ \frac{1}{\epsilon} + \frac{1}{2} \ln |u| \right] . \quad (1.20)$$

The net result for the imaginary part is then:

$$\text{Im}_s [I(s, t) + I(s, u) + I(u, t)] = N^2 \pi c(\epsilon) \left[ \frac{1}{\epsilon} \frac{1}{tu} - \frac{1}{2st} \ln |t| - \frac{1}{2su} \ln |u| \right] . \quad (1.21)$$

This simple analysis shows that, depending on whether  $|t|$  and  $|u|$  is greater or lesser than one, we may change the sign of the imaginary part. The value of  $\epsilon$  also enters into the above infra-red divergent integral and may change its sign; we shall take into account the interchange of limits as we examine the unitarity relation between weak and strong 't Hooft coupling.

As we take  $k_1 = k_4$  and  $k_2 = k_3$ , we force the Mandelstam invariants to the limit  $t = 0$  and  $-u = s = x$ . The limiting form of the imaginary part in this case is

$$\text{Im}_s [I(s, t) + I(s, u) + I(u, t)] = -N^2 \pi c(\epsilon) \frac{1}{tx} \left[ \frac{1}{\epsilon} + \frac{1}{2} \ln |t| \right] + N^2 \pi c(\epsilon) \frac{1}{2x^2} \ln |x| , \quad (1.22)$$

and is apparently negative for fixed  $x$  and  $t \rightarrow 0^-$  for fixed  $\epsilon$  very small. For  $t$  infinitesimally small, the value is positive or negative depending on the value of  $\epsilon$ . The full dimensionally regularized expression, that is the result for non-zero  $\epsilon$  at  $t = 0$ , is non-vanishing but does not possess an imaginary part (For example, the integral  $I(t, u)$  reduces to  $(-u)^{-2-\epsilon}$  times a constant and is purely real. Remaining integrals are similarly evaluated.)

The preceding calculation in the tree-level IIB superstring theory on  $AdS_5 \times S^5$  in a Ramond-Ramond background always gave a positive value. In the case of external string states in our amplitude calculation only internal propagating string modes contribute, but the analysis may be performed for general four-point correlation functions. The calculation of the imaginary parts arising from the string modes in the AdS space translates into evaluating the large  $N$ , finite  $\lambda$ , imaginary part of the holographic  $S$ -matrix element.

A conundrum arises when we compare between the large  $\lambda$  and small  $\lambda$  regimes. From explicitly evaluating the two different integrals we conclude that the imaginary part must pass through zero, or becomes zero at the value  $\lambda = 0$  within dimensional regularization. Define this critical value of the coupling as  $\lambda_0(k_j)$ . At a finite value of the coupling, such as  $\lambda_0$ , only the positive functions, the squares in (1.1), contribute. This mismatch indicates a finite radius of convergence of the perturbative series defined either by the  $N = 4$  field theory at large  $N$ , or at holographic string tree-level. Hence naively the two regions do not agree with each other.<sup>1</sup>

There are several options to account for this disagreement: (1) There is a phase transition at finite  $\lambda$  and at large  $N$  in the dual theory to IIB superstring theory reflecting the divergence of the tree-level holographic scattering element (related to [25]); (2) Related is that the series at infinite  $N$  has a finite radius of convergence in  $1/\lambda$  from  $\lambda$  large: there is either a pole on the positive real axis of  $\lambda$  or a cut passing through or laying on top of it (possibly also in a Borel transform reflecting a finite radius of convergence); (3) Unitarity is not preserved in the holographic  $AdS$  formulation; (4) There are modifications yet to be specified in the matching between  $N = 4$  super Yang-Mills theory and IIB superstring theory on  $AdS_5 \times S^5$  at finite  $\lambda$ , away from  $\lambda = \infty$ .

Instanton-like effects may be predicted in the case of option (1) or (2) similar to the  $e^{-1/g_s}$  effects predicted in the analysis of large orders of perturbative string theory to restore convergence; likewise, within the correspondence we may infer similar contributions missed in  $N = 4$  super Yang-Mills theory which, unlike the 't Hooft instanton, contribute in the infinite  $N$  limit and are not easily seen with the use of field equations. (It is interesting that in unbroken  $N = 4$  super Yang-Mills theory the BPS mass formula indicates an infinite number of massless dyons in the theory.) The existence of further contributions is supported by the evidence for finite radii of convergence in perturbative expansions in the large  $N$  limit [26] and evidence for potential non-Borel summability in various field theories. If present, these effects would deserve dual  $AdS$  descriptions possibly via the inclusion of additional states

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<sup>1</sup>It is possible that this comparison between the field theory and string theory is regularization dependent, but the string theory obeys the factorization in (1.1) also in the radial coordinate regulating scheme  $z_0 > r_c$  in addition with dimensional reduction. The two types of delta function terms, those proportional to finite or infinite coefficients [22], are not encountered directly here because we are working in momentum space and have not examined possible AdS boundary term effects.



or processes within the string theory side (and contributing at intermediate coupling in  $\lambda$ ). Similar behavior is found in the context of perturbative string theory [23, 24] and transitions have been argued for in  $N = 4$  super Yang-Mills theory at finite temperature [27]. Additional modifications would have to leave the two- and three-point functions of chiral primary operators unchanged at leading order in  $N$  to maintain the validity of previous work regarding the matching [18, 19, 20, 21].

Another option is that the higher-order terms in the  $\alpha'$  expansion of the non-abelian Dirac-Born-Infeld action are to be included in the super Yang-Mills theory description. This option would suggest that  $N = 4$  super Yang-Mills theory is only dual to IIB string theory on AdS at  $\lambda = \infty$ . Sub-leading effects would necessitate these higher order terms in the DBI expansion. However, this possibility is problematic in that the DBI action is non-renormalizable in four dimensions and further inclusion of terms might alter the two- and three-point correlation functions at sub-leading order.

While this work was being completed, an analysis of the effects of finite radii of convergence in an AdS/CFT correspondence have been presented in the context of non-supersymmetric type 0B theory [28].

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